Dynamical behavior of Lagrangian systems on Finsler manifolds

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In this paper we develop a theoretical framework devoted to a geometrical description of the behavior of dynamical systems and their chaotic properties. The underground manifold is a Finsler space whose features permit the description of a wide class of dynamical systems such as those with potentials depending on the time and velocities for which the Riemannian approach is unsuitable. Another appealing feature of this more general setting relies on its very origin: Finsler spaces arise in a direct way on imposing the invariance for time reparametrization to a standard variational problem. A Finsler metric is a generalization of the well-known Jacobi and Eisenhart-metrics for conservative dynamical systems. We use this geometry to derive the main geometrical invariants and related expressions that are needed to establish the transition to chaos in very general Lagrangian systems. In order to point out the versatility and the effectiveness of this extension of the geometrical approach, we suggest the introduction of this formalism to some interesting dynamical systems for which the Finsler metric is much more suitable than the Riemannian one. In particular, we present the following: (i) an exhaustive description and numerical results for a resonant oscillator with a time-dependent potential, (ii) an exact description (without any approximation) of the dynamics of Bianchi type-IX cosmological models, and (iii) a geometrical description of the restricted three-body problem whose effective potential depends linearly on the velocities. In the first case, the numerical integration of the geodesics and geodesic deviation equations shows that in the geometrical picture the source of the exponential instability of trajectories relies on the mechanism of parametric resonance and does not originate from the negativity of curvature.
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I. INTRODUCTION

The goal of understanding the qualitative behavior of dynamical systems has often been pursued by studying the properties of a corresponding geodesic flow obtained by a suitable ''geometrization'' procedure [1]. However, while many rigorous results concerning the global picture of the motion have been obtained in the case of systems whose geodesic-flows involve strictly negative curvature, identifying also their statistical properties [2] there are few results for generic systems with non-negative curvature. The attempts made in these cases (see [3,4] and references therein) have been based on the study of (exact or approximate) forms of the geodesic deviation equation, trying to establish the link between the long-time behavior of trajectories and some suitable averages of geometric invariants. Despite the fact that these results are yet not complete and a rigorous setting is still being sought, the geometric approach is promising, however, and deserves more attention.

The purpose of this paper is to illustrate (also through some *realistic* physical applications) a more general geometrization procedure based on the study of geodesic flows over *Finsler manifolds* [5]. The need of a generalization of the Riemannian approach to chaos emerges when the dynamical structure differs from the ''standard'' one, as in the case of non-Hamiltonian systems or when there is a not purely quadratic dependence on velocities or momenta. The interesting aspect of the application of Finsler geometry is that it allows the treatment of a wider class of Lagrangian dynamical systems, such as those with potentials depending on time, with gyroscopic (i.e., velocity-dependent) terms and without positive-definite kinetic energy. The Finsler metric is a generalization of the well-known Jacobi metric, used to reduce the motion of a conservative system to geodesic motion over a conformally Euclidean manifold [3,4,6]. The greater generality of the Finsler metric is due to the fact that the metric tensor depends on velocities as well as coordinates and, despite the drawback of having more involved expressions for geometric invariants (the metric is in general not diagonal and the components of the connection also depend on velocities), it is not difficult to obtain a description of the qualitative behavior of a generic Lagrangian system by exploring the stability of the solutions of the geodesic deviation equation.

The actual starting point for this survey was the attempt towards a successful geometrization of Bianchi IX cosmology, to investigate the very nature of its approach to the singularity, the character of which has been controversial for quite some time [7]. It is generally accepted that the sources of these controversial results lie in the gauge freedom imposed by the general relativity and consequently on the time variable chosen. There have already been several attempts towards a geometric or intrinsic characterization of the dy-

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namics, and more recently other geometrical approaches [8] have been applied, essentially using the Jacobi geometry, but all these are subject to conceptual and technical shortcomings that do not allow completely reliable predictions [9,10,5(b)]. As shown below, the use of Finsler geometry by passes the problems caused by the singularity in the kinetic-energy term of the Arnowitt-Deser-Misner Hamiltonian [11], thus removing the main source of trouble with the Jacobi metric. Moreover, it is just in the framework of general relativistic theories that the Finslerian description of dynamics reveals its major merits. Indeed, Finsler spaces naturally arise when the invariance for reparametrization of time is imposed on a standard variational problem, this ''built-in'' invariance makes the use of Finsler *geometrodynamics* the best choice to cope with such theories.

Besides this, the range of applicability of the Finslerian approach is wider than what the peculiarity of Hamiltonian cosmologies requires. It can, in fact, be used to analyze time-dependent systems, leading in a natural way to a geodesic flow over an extended configuration manifold: As an example of this application we illustrate below the simple but paradigmatic case of the time-varying frequency harmonic oscillator. However, it is quite straightforward to study gyroscopic Lagrangians, such as that associated with the restricted three-body problem in the rotating coordinate system.

The plan of the paper is as follows. In Sec. II we give a brief overview of Finsler geometry. In Sec. III we present the relation between Lagrangian analytical mechanics and geodesic flows over Finslerian manifolds. In Sec. IV we discuss the applications mentioned above. In Sec. V we conclude with the presentation of forthcoming results and possible developments.

II. FINSLER GEOMETRY

In this section we briefly describe the main properties of Finslerian-manifolds [5], which can be considered a generalization of Riemannian spaces because of the dependence of the line element on both the coordinates x^i and velocities $x'^i = dx^i/dw$, where w is an arbitrary time parameter. The line element is defined as

$$ds_F^2 = \Lambda^2(x^i, x'^i)dw^2 = g_{ij}(x^k, x'^k)dx^i dx^j,$$

$$g_{ij} = \frac{1}{2}\frac{\partial^2 \Lambda^2(x^k, x'^k)}{\partial x'^i \partial x'^j}. \tag{1}$$

Note that if $\Lambda^2(x^k, x'^k)dw^2 = g_{ij}(x^k)dx^i dx^j$, the manifold reduces to a Riemannian one. The function $\Lambda(x^i, x'^i)$ must satisfy the conditions:

$$\Lambda(x^i, kx'^i) = k\Lambda(x^i, x'^i), \quad k>0, \tag{2}$$

i.e., $\Lambda(x^i, x'^i)$ is a positively homogeneous function of degree 1 in x'^i,

$$\Lambda(x^i, x'^i) \neq 0, \quad \forall \quad x'^i \neq 0, \tag{3}$$

and, moreover, if a sign definite metric is required,

$$\frac{\partial^2 \Lambda^2(x^k, x'^k)}{\partial x'^i \partial x'^j}\xi^i \xi^j > 0, \quad \forall \xi^i \neq \lambda x'^i. \tag{4}$$

The geodesic equations, formally analogous to the Riemannian ones, written in terms of the conformal parameter s are

$$x''^j + \gamma^j_{hk}(x^i, x'^i)x'^h x'^k = 0, \tag{5}$$

where

$$\gamma^j_{hk}(x^l, x'^l) \overset{\text{def}}{=} \frac{1}{2} g^{ij}\left(\frac{\partial g_{ih}}{\partial x^k} + \frac{\partial g_{ik}}{\partial x^h} - \frac{\partial g_{kh}}{\partial x^i}\right) \tag{6}$$

are the generalized connections. The geodesic deviation equations [5] only formally analogous to the Riemannian ones, are given by

$$\frac{\delta^2 z^i}{\delta s^2} + K^i_{jhk}(x^i, x'^i)x'^j x'^h z^k = 0, \tag{7}$$

where $\delta/\delta s$ is the so-called delta differentiation

$$\frac{\delta z^i}{\delta s} = \frac{dz^i}{ds} + \gamma^i_{hj}x'^h z^j \tag{8}$$

and $K^i_{jhk}(x^i, x'^i)$ is one of the various curvature tensors that can be defined in a Finsler manifold. Its expression contains terms involving the derivatives with respect to the velocities in addition to the usual terms in the Riemann tensor. It is defined by

$$K^i_{jhk}(x^i, x'^i) \overset{\text{def}}{=} \left(\frac{\partial \Gamma^{*i}_{jh}}{\partial x^k} - \frac{\partial \Gamma^{*i}_{jh}}{\partial x'^l}\frac{\partial G^l}{\partial x'^k}\right) - \left(\frac{\partial \Gamma^{*i}_{jk}}{\partial x^h} - \frac{\partial \Gamma^{*i}_{jk}}{\partial x'^l}\frac{\partial G^l}{\partial x'^h}\right) + \Gamma^{*i}_{mk}\Gamma^{*m}_{jh} - \Gamma^{*i}_{mh}\Gamma^{*m}_{jk}, \tag{9}$$

where

$$G^i \overset{\text{def}}{=} \tfrac{1}{2}\gamma^i_{jk}x'^j x'^k, \tag{10}$$

$$\Gamma^{*h}_{kj} \overset{\text{def}}{=} \gamma^h_{kj} - g^{ih}\left(C_{jil}\frac{\partial G^l}{\partial x'^k} + C_{kil}\frac{\partial G^l}{\partial x'^j} - C_{kjl}\frac{\partial G^l}{\partial x'^i}\right), \tag{11}$$

$$C_{ijk} \overset{\text{def}}{=} \frac{1}{2}\frac{\partial g_{ij}}{\partial x'^k}. \tag{12}$$

In the Riemannian case $g_{ij} = g_{ij}(x^i)$ and the curvature tensor K^i_{jhk} reduces to the Riemann tensor.

As in Riemannian geometry [3], we define a *stability tensor*

$$H^i{}_k \overset{\text{def}}{=} K^i_{jhk}x'^j x'^h. \tag{13}$$

As shown below, this tensor contains all the information about the dynamical behavior of the system and in particular determines the *possible* chaotic properties.

III. DYNAMICS ON FINSLER MANIFOLD

The Finsler geometry is particularly suited to describe Lagrangian systems for which it provides the tangential space with a metric [5]. In the analysis of a dynamical system, $\Lambda(x^i,x'^i)$ is assumed to be the homogeneous Lagrangian deriving from the standard Lagrangian $L(t=x^0,x^\alpha,dx^\alpha/dt)$ $(\alpha=1,\ldots,n)$ via the correspondence

$$L\to\Lambda=L\left(x^i,\frac{x'^\alpha}{x'^0}\right)\cdot x'^0,\quad x'^i=\frac{dx^i}{dw}$$
$$(i=0,\ldots,n;\ \ \alpha=1,\ldots,n),\tag{14}$$

leading to a homogeneous function of first degree in the velocities x'^i. Thus, for a dynamical system with n degrees of freedom, the corresponding Finsler space is an $(n+1)$-dimensional manifold. Note that $\Lambda(x^i,dx^i)$ is invariant under a rescaling of the *time* parameter w. In fact, if w_1 and w_2 are two different parameters, then due to the property (2) it follows that

$$\Lambda\left(x^i,\frac{dx^i}{dw_1}\right)dw_1=\Lambda\left(x^i,\frac{dx^i}{dw_2}\right)dw_2.\tag{15}$$

The $n+1$ Euler-Lagrange equations for the homogeneous Lagrangian are equivalent to the n equations of motion and an additional one involving the time derivative of the energy.

From Eq. (1) it follows that the line element is

$$ds=\Lambda\left(x^i,\frac{dx^i}{dw}\right)dw,\tag{16}$$

so that if we make the conformal choice of time parameter $w=s$, then

$$\frac{ds}{dw}=\Lambda\left(x^i,\frac{dx^i}{dw}\right)\equiv 1\tag{17}$$

and $x'^0=1/L$. In fact,

$$(x'^0)^{-1}=\frac{ds}{dt}=\Lambda\left(x^i,\frac{dx^i}{dt}\right)=L\left(x^i,\frac{dx^\alpha}{dt}\right).\tag{18}$$

In order to study the dynamical behavior of a wide class of n-dimensional systems, we specialize our analysis to the systems described by the Lagrangians

$$L=\mathcal{T}-\sum_\alpha f_\alpha(x^i)\dot{x}^\alpha-U(x^i),\quad \alpha=1,\ldots,n;\quad i=0,\ldots,n,\tag{19}$$

where $\dot{x}^\alpha=dx^\alpha/dt$ and

$$\mathcal{T}=\frac{1}{2}\sum_\alpha a_\alpha(\dot{x}^\alpha)^2,\tag{20}$$

where a_α are real constants and both the potential $U=U(x^i)$ and the functions $f_\alpha=f_\alpha(x^i)$ depend in general only on the spatial coordinates x^α and time $t=x^0$. Moreover, we assume coordinates in which the kinetic-energy matrix is diagonal. In contrast to Sec. II, the summation convention is not assumed here. The above class of Lagrangians is very general and includes conservative systems, systems with a potential depending on the time and/or on velocities, and even systems with a Lorentz signature in the kinetic part like the ones coming from general relativity (see, e.g., Sec. IV B below on Bianchi IX cosmological models). From Eqs. (14) and (19), one obtains the homogeneous Lagrangian

$$\Lambda=\frac{T}{x'^0}-\sum_\alpha f_\alpha\cdot x'^\alpha-U\cdot x'^0,\tag{21}$$

where

$$T=\frac{1}{2}\sum_\alpha a_\alpha(x'^\alpha)^2=\mathcal{T}(x'^0)^2.\tag{22}$$

A. Deriving the Finsler metric

In order to derive the Finsler metric Λ or, equivalently, the Lagrangian L must be sign definite. If this is not the case, one can make it so by adding a constant to the potential or a gauge function to the Lagrangian L

$$\tilde{L}=L+\frac{d\mathcal{G}(x^i)}{dt}\tag{23}$$

since the equations of motion are the same for L and L'.

So, starting from a Lagrangian satisfying conditions (2)–(4), the covariant components of the Finsler metric are easily evaluated using the definition (1),

$$g_{00}=\frac{3T^2}{(x'^0)^4}+U^2-\frac{2T}{(x'^0)^3}\sum_\alpha f_\alpha x'^\alpha,\tag{24}$$

$$g_{\alpha\alpha}=\left(a_\alpha\frac{x'^\alpha}{x'^0}-f_\alpha\right)^2+a_\alpha\frac{\Lambda}{x'^0},\tag{25}$$

$$g_{0\alpha}=-\left(\frac{T}{(x'^0)^2}+U\right)\left(a_\alpha\frac{x'^\alpha}{x'^0}-f_\alpha\right)-a_\alpha\frac{x'^\alpha}{(x'^0)^2}\Lambda,\tag{26}$$

$$g_{\alpha\beta}=\left(a_\alpha\frac{x'^\alpha}{x'^0}-f_\alpha\right)\left(a_\beta\frac{x'^\beta}{x'^0}-f_\beta\right).\tag{27}$$

The determinant of the metric g_{ij} is

$$g=\frac{\Lambda^{n+2}}{(x'^0)^{n+2}}\prod_\alpha a_\alpha.\tag{28}$$

The contravariant components are

$$g^{00}=\frac{(x'^0)^2}{\Lambda^3}\left[3\Lambda+2Ux'^0+x'^0\sum_\alpha\frac{f_\alpha^2}{a_\alpha}\right],\tag{29}$$

$$g^{\alpha\alpha}=\frac{x'^0}{\Lambda^3}\left[\frac{1}{a_\alpha}\left(\frac{T}{x'^0}-Ux'^0\right)^2+(x'^\alpha)^2\left(\frac{\Lambda}{x'^0}+2U\right)\right]+\frac{2x'^0}{a_\alpha\Lambda^3}\sum_{\beta\neq\alpha}\left[\frac{f_\beta^2T}{a_\beta}-f_\beta x'^\beta\left(\frac{T}{x'^0}-Ux'^0\right)\right],\tag{30}$$

$$g^{0\alpha}=\frac{x'^0}{\Lambda^3}\cdot\left[2x'^\alpha(\Lambda+Ux'^0)+x'^0\frac{f_\alpha}{a_\alpha}\Lambda+x'^0x'^\alpha\cdot\sum_\alpha\frac{f_\alpha^2}{a_\alpha}\right], \tag{31}$$

$$g^{\alpha\beta}=\frac{x'^0}{\Lambda^3}\cdot\left[x'^\alpha x'^\beta\left(\frac{\Lambda}{x'^0}+2U\right)-2T\frac{f_\alpha}{a_\alpha}\frac{f_\beta}{a_\beta}+\left(\frac{T}{x'^0}-Ux'^0\right)\times\left(\frac{f_\alpha}{a_\alpha}x'^\beta+\frac{f_\beta}{a_\beta}x'^\alpha\right)\right]. \tag{32}$$

Note that, in contrast to the Jacobi and Eisenhart metrics, the Finsler metric is not diagonal even for conservative systems.

With this metric the equations of motion are reduced to a geodesic flow on a Finsler manifold. Since we are interested in behavior along geodesics, in what follows we use the conformal time parameter s so that $x'^i=dx^i/ds$ and $\Lambda\equiv1$.

B. The stability tensor

The stability tensor defined by Eq. (13) and all the geometrical quantities that are involved in the geodesic deviation equation have been defined in order to study the dynamical behavior of a Lagrangian system. We define the expressions

$$A\overset{\text{def}}{=}-\frac{1}{2}\frac{dL}{ds}, \tag{33}$$

$$B\overset{\text{def}}{=}\frac{dA}{ds}+x'^0A^2 \tag{34}$$

and notation

$$Q_{,i}\overset{\text{def}}{=}\frac{\partial Q}{\partial x^i},\quad Q_{,\prime i}\overset{\text{def}}{=}\frac{\partial Q}{\partial x'^i},\quad Q_{,ij}\overset{\text{def}}{=}\frac{\partial^2 Q}{\partial x^i\partial x^j}; \tag{35}$$

$$(\nabla^*Q)^2\overset{\text{def}}{=}\sum_\alpha\frac{(Q_{,\alpha})^2}{a_\alpha},\quad \Delta^*Q\overset{\text{def}}{=}\sum_\alpha\frac{Q_{,\alpha\alpha}}{a_\alpha}. \tag{36}$$

The components of the stability tensor defined by Eq. (13) are

$$H^0{}_0=-3(x'^0)^2A^2+Bx'^0\left(1+\frac{2T}{x'^0}-\sum_\alpha f_\alpha x'^\alpha\right)+(x'^0)^2\times\left(\frac{dA_{,\prime 0}}{ds}-2A_{,0}\right)+(x'^0)^2\sum_\alpha\frac{A_{,\prime\alpha}}{a_\alpha}\left[x'^0U_{,\alpha}-\frac{x'^0}{2}f_{\alpha,0}-\frac{1}{2}\frac{df_\alpha}{ds}+\frac{1}{2}\sum_\gamma f_{\gamma,\alpha}x'^\gamma\right], \tag{37}$$

$$H^\alpha{}_\alpha=(x'^0)^2\frac{U_{,\alpha\alpha}}{a_\alpha}+Bx'^0\left(1+x'^\alpha f_\alpha-\frac{(x'^\alpha)^2}{x'^0}a_\alpha\right)+x'^0x'^\alpha\left(\frac{dA_{,\prime\alpha}}{ds}-2A_{,\alpha}\right)+\frac{(x'^0)^2x'^\alpha}{2}\sum_{\beta\neq\alpha}\frac{A_{,\prime\beta}}{a_\beta}(f_{\alpha,\beta}-f_{\beta,\alpha})+\frac{(x'^0)^2}{4a_\alpha}\sum_{\beta\neq\alpha}\frac{1}{a_\beta}(f_{\alpha,\beta}-f_{\beta,\alpha})^2-\frac{x'^0}{a_\alpha}\frac{df_{\alpha,\alpha}}{ds}+\frac{x'^0}{a_\alpha}\sum_\beta f_{\beta,\alpha\alpha}x'^\alpha, \tag{38}$$

$$H^0{}_\alpha=Bx'^0(x'^0f_\alpha-x'^\alpha a_\alpha)+(x'^0)^2\left(\frac{dA_{,\prime\alpha}}{ds}-2A_{,\alpha}\right)+\frac{(x'^0)^3}{2}\sum_{\beta\neq\alpha}\frac{A_{,\prime\beta}}{a_\beta}(f_{\alpha,\beta}-f_{\beta,\alpha}), \tag{39}$$

$$\begin{aligned}H^\alpha{}_0={}&(x'^0)^2\frac{U_{,0\alpha}}{a_\alpha}-3x'^0x'^\alpha A^2+Bx'^\alpha\left(\frac{2T}{x'^0}-\sum_\beta f_\beta x'^\beta\right)+x'^0x'^\alpha\left(\frac{dA_{,\prime 0}}{ds}-2A_{,0}\right)+\frac{x'^0}{a_\alpha}\left[\sum_\beta(f_{\beta,0\alpha}x'^\beta)-\frac{dU_{,\alpha}}{ds}\right]\\&+\frac{1}{2a_\alpha}\sum_\beta x'^\beta\left(\frac{df_{\alpha,\beta}}{ds}-\frac{df_{\beta,\alpha}}{ds}\right)-\frac{x'^0x'^\alpha}{4a_\alpha}\sum_\beta\frac{1}{a_\beta}(f_{\alpha,\beta}-f_{\beta,\alpha})^2\\&+x'^0x'^\alpha\left\{\sum_\beta\frac{A_{,\prime\beta}}{a_\beta}\left[U_{,\beta}x'^0-\frac{x'^0}{2}f_{\beta,0}-\frac{1}{2}\frac{df_\beta}{ds}+\frac{1}{2}\sum_\gamma(f_{\gamma,\beta}x'^\gamma)\right]\right\},\end{aligned} \tag{40}$$

$$H^\alpha{}_\beta=(x'^0)^2\frac{U_{,\alpha\beta}}{a_\alpha}+Bx'^\alpha(x'^0f_\beta-x'^\beta a_\beta)+x'^0x'^\alpha\times\left(\frac{dA_{,\prime\beta}}{ds}-2A_{,\beta}\right)-\frac{(x'^0)^2x'^\alpha}{2}\frac{A_{,\prime\alpha}}{a_\alpha}(f_{\alpha,\beta}-f_{\beta,\alpha})+\frac{x'^0}{a_\alpha}\left[\sum_\gamma f_{\gamma,\alpha\beta}x'^\gamma-\frac{1}{2}\left(\frac{df_{\alpha,\beta}}{ds}+\frac{df_{\beta,\alpha}}{ds}\right)\right]. \tag{41}$$

Because the antisymmetry property $K^i_{jhk}=-K^i_{jkh}$ of the curvature tensor, the determinant $\det(H^i{}_j)$ vanishes and

$$H^i{}_jx'^j=0, \tag{42}$$

as expected, so that x'^J is an eigenvector of $H^i{}_j$ with an eigenvalue zero. The trace ($\equiv H^\iota{}_\iota$) of the stability tensor is one of the synthetic indicators of instability [3,4] involved in the geodesic deviation equation for the norm of the perturbation and it is useful when the dynamical system under consideration has many degrees of freedom. As in Riemannian geometry, the trace is the *Ricci curvature* along the geodesic flow. Indeed,

$$\mathrm{Ric}(a^i) \stackrel{\mathrm{def}}{=} K^i_{jki} a^j a^k \tag{43}$$

(see, e.g., [5(a)], p. 131) and if $a^i \equiv x'^i$, then $\mathrm{Ric}(a^i) \equiv H^i{}_i$ [3]. Its formula is

$$H^i{}_i = n x'^0 B + \frac{(x'^0)^2}{4} \sum_\alpha \sum_{\beta \neq \alpha} \frac{(f_{\alpha,\beta} - f_{\beta,\alpha})^2}{a_\alpha a_\beta} + (x'^0)^2 \Delta^* U + x'^0 \sum_\alpha x'^\alpha \Delta^* f_\alpha - x'^0 \sum_\alpha \frac{1}{a_\alpha} \frac{d f_{\alpha,\alpha}}{ds}. \tag{44}$$

The other meaningful geometrical stability indicators are the eigenvalues $\lambda_{(A)}$ of the stability tensor, where $A = 0,1,\ldots,n$ label different eigenvectors. As in Riemannian geometry, each eigenvalue is the sectional curvature of the two-surface determined by the tangent to the flow and the normalized eigenvector $X^i_{(A)}$ of $H^i{}_j$ associated with $\lambda_{(A)}$. In fact,

$$K^{(2)}(x'^i, X^i_{(A)}) = K_{jihk} x'^j x'^h X^i_{(A)} X^k_{(A)} = g_{il} H^l{}_k X^l_{(A)} X^k_{(A)} = \lambda_{(A)}. \tag{45}$$

The eigenvalue corresponding to the eigenvector along the geodesic flow is zero $\lambda_{(0)} = 0$, while the others can be expressed in the form

$$\lambda_{(\nu)} = x'^0 B + \frac{(x'^0)^2}{4n} \sum_\alpha \sum_{\beta \neq \alpha} \frac{(f_{\alpha,\beta} - f_{\beta,\alpha})^2}{a_\alpha a_\beta} + x'^0 \sigma_{(\nu)}, \tag{46}$$

where $\sigma_{(\nu)}$ are the eigenvalues of a matrix $\mathcal{D}^\alpha{}_\beta$, whose elements are

$$\mathcal{D}^\alpha{}_\beta = \frac{1}{a_\alpha} \cdot \left[U_{,\alpha\beta} x'^0 + \sum_\gamma f_{\gamma,\alpha\beta} x'^\gamma - \frac{1}{2} \cdot \left(\frac{d f_{\alpha,\beta}}{ds} + \frac{d f_{\beta,\alpha}}{ds} \right) \right]. \tag{47}$$

The matrix $\mathcal{D}^\alpha{}_\beta$ is equivalent to the Hessian matrix of the potential, involved in the tangent dynamics. In the conservative case, with a standard kinetic form, since the Lagrangian does not depend on time, $a_\alpha = 1$ and $f_\alpha = 0$, $D^\alpha{}_\beta$ is just the Hessian matrix of the potential $U(x^\alpha)$, apart from a factor x'^0. Finally, we list the covariant components of the stability tensor in a Finsler space $H_{ij} \equiv H_{ji}$ [5],

$$H_{00} = \sum_\alpha f_{\alpha,00} x'^\alpha + \frac{2T}{(x'^0)^2} B + \frac{1}{x'^0} \sum_{\alpha,\beta} U_{,\alpha\beta} (x'^\alpha)^2 - \sum_\alpha \frac{x'^\alpha}{x'^0} \frac{d f_{\alpha,0}}{ds} + \frac{T}{2x'^0} \sum_\alpha \sum_{\beta \neq \alpha} \frac{(f_{\alpha,\beta} - f_{\beta,\alpha})^2}{a_\alpha a_\beta}, \tag{48}$$

$$H_{\alpha\alpha} = x'^0 U_{,\alpha\alpha} + \sum_\beta f_{\beta,\alpha\alpha} x'^\beta + a_\alpha B - \frac{d f_{\alpha,\alpha}}{ds} + \frac{x'^0}{4} \sum_{\beta \neq \alpha} \frac{(f_{\alpha,\beta} - f_{\beta,\alpha})^2}{a_\beta}, \tag{49}$$

$$H_{0\alpha} = x'^0 U_{,0\alpha} + \sum_\beta f_{\beta,0\alpha} x'^\beta - a_\alpha \frac{x'^\alpha}{x'^0} B - \frac{d U_{,\alpha}}{ds} + \frac{1}{2x'^0} \sum_{\beta \neq \alpha} x'^\beta \left(\frac{d f_{\alpha,\beta}}{ds} - \frac{d f_{\beta,\alpha}}{ds} \right) - \frac{x'^\alpha}{4} \sum_{\beta \neq \alpha} \frac{(f_{\alpha,\beta} - f_{\beta,\alpha})^2}{a_\beta}, \tag{50}$$

$$H_{\alpha\beta} = x'^0 U_{,\alpha\beta} + \sum_\gamma f_{\gamma,\alpha\beta} x'^\gamma - \frac{1}{2} \left(\frac{d f_{\alpha,\beta}}{ds} + \frac{d f_{\beta,\alpha}}{ds} \right). \tag{51}$$

C. Discussion

The instability properties of a dynamical system are described by the geodesic deviation equation (7). In analogy with the Lyapunov exponent (see, e.g., [12]), we define an *instability* exponent given by

$$\delta_I = \lim_{s \to \infty} \lim_{z(0) \to 0} \left[\frac{1}{s} \ln \frac{z(s)}{z(0)} \right], \tag{52}$$

where $z(s) \equiv \| z^i(s) \| = \sqrt{g_{ij} z^i z^j}$ is the norm of the perturbation with respect to the Finsler metric. The exponent δ_I is a measure of the asymptotic growth rate of the perturbation of a given geodesic. An alternative approach is to calculate n instability exponents, one associated with each component of the perturbation z^i in a basis chosen normal to the geodesic flow, since the tangential component increases almost linearly [see Eq. (42)] with the time s. If there is an exponential divergence of nearby geodesics, at least one of these components exhibits a positive instability exponent.

When the system has many degrees of freedom $n \gg 1$, one approach is to analyze an approximate version of the geodesic deviation equation

$$\frac{d^2 z}{ds^2} + \frac{H^i{}_i}{n} z = 0 \tag{53}$$

involving the trace of the stability tensor [3,4] and the ordinary derivative of the norm z of the perturbation. When the system has a few degrees of freedom, it is necessary to study the full system of equations in which z^i is expressed in a particular basis, since the averaging procedure implied by Eq. (53) can mask the sources of instability.

IV. APPLICATIONS AND EXAMPLES

In this section we show some paradigmatic examples to which we apply the Finsler geometrical description. In the first example we discuss the instability behavior of a resonant oscillator whose features are well known, so that it is possible to compare the results of our method with those obtained by analytical approximations of the tangent dynamics. Although this is a very simple example, nevertheless, its purpose is to show how we can get insights into the sources of instability (and possibly of chaos) in the geometric framework. In the last part of this section we give a geometrical description of the dynamics of the Bianchi IX cosmological models and, finally, of the restricted three-body problem. We show that in both cases the Finsler geometry is particularly suited to the description of the dynamical properties and allows one to attach an intrinsic meaning to the results.

Let us now compare the thoroughly studied Jacobi metric with the Finsler metric in the case of conservative systems with n degrees of freedom. At variance with the Jacobi metric in which the conformal factor is the kinetic energy, in the Finsler metric it is given by the Lagrangian. As it has been shown in [3,4], the evolution of (the norm of) a perturbation to a geodesic in the Jacobi metric, for a system with n degrees of freedom, is governed in a first approximation (which becomes better and better as n increases) by the Ricci curvature *along the flow*, given by

$$H_J=\frac{1}{2W^2}\left\{\Delta U+\frac{(\nabla U)^2}{W}+(n-2)\right.$$
$$\left.\times\left[\frac{1}{2}\left(\frac{dU}{ds_J}\right)^2+W\,\frac{d^2U}{ds_J^2}\right]\right\}, \tag{54}$$

where $W\equiv E-U$ is the kinetic energy (E and U being the total and the potential energies, respectively). Analogously, the synthetic indicator of stability for a Finsler geodesic is the trace of $H^i{}_j$:

$$\mathrm{tr}\ H_F=t'^2\Delta U+nt'\,\frac{d^2U}{ds^2}+nt'^2\left(\frac{dU}{ds}\right)^2, \tag{55}$$

where $t'=1/L$ is the inverse of the Lagrangian. The differences between the trace of H_F and the trace of H_J consist in the absence of the gradient and, more relevant, in the presence of the Lagrangian instead of the kinetic energy. Owing to the gauge freedom in the definition of a Lagrangian, which allows one to make it always sign definite (i.e., never vanishing), the Finsler metric is well suited in the case of a few degrees of freedom dynamical systems and even more when the system is near integrability, as in both cases the possibility that the kinetic energy vanishes giving a singularity in the conformal factor of the Jacobi metric is not negligible; this can be avoided working in a Finsler manifold.

A. Resonant oscillator and numerical results

As a first application, we study a one-dimensional system with time-dependent potential

$$U(x,t)=\tfrac{1}{2}\omega^2(t)x^2. \tag{56}$$

Though this is a fairly simple example whose behavior can be determined with high accuracy by analytical approximations [13], nevertheless, it allows us to understand the mechanism that governs the exponential instability. As it has been also found in other ''realistic'' cases of interest [4,5(b),10,14–16], in the corresponding geometric picture of dynamics instability does not originate from the negativity of the curvatures, as it has been argued before, but is generally caused by a parametric resonance due to the rapid fluctuations of positive curvatures.

For the system we are discussing, the Finsler manifold is a two-dimensional manifold whose metric is derived from the homogeneous Lagrangian

$$\Lambda=\frac{(x')^2}{2t'}-U(x,t)t'. \tag{57}$$

The covariant components of the metric are

$$g_{tt}=\frac{3}{4}\,\frac{(x')^4}{(t')^4}+U^2,$$
$$g_{xx}=\frac{3}{2}\,\frac{(x')^2}{(t')^2}-U,$$
$$g_{tx}=-\frac{(x')^3}{(t')^3}. \tag{58}$$

Because we are dealing with a two-dimensional manifold, the geodesic deviation equation reduces to

$$\frac{d^2z_\parallel}{ds^2}=0, \tag{59a}$$

$$\frac{d^2z_\perp}{ds^2}+\lambda z_\perp=0, \tag{59b}$$

where the sectional curvature λ of the two-dimensional surface determined by the direction of the geodesic flow and by a normal vector is now the only nonvanishing eigenvalue of the stability tensor and has the expression

$$\lambda=Bt'+(t')^2U_{,xx}=t'U_{,xx}[t'+(x')^2]+\frac{(t')^3}{2}\,U_{,tt}$$
$$+\frac{3}{2}\,U_{,tx}(t')^2x'+(t')^2U_{,x}{}^2(3(x')^2-t')+\frac{3}{4}\,(t')^4U_{,t}{}^2$$
$$+3(t')^3x'U_{,x}U_{,t}\,. \tag{60}$$

The possible instability behavior of the system is described by the second of equations (59), giving the evolution of a perturbation normal to the geodesic flow.

Let us now consider the expression for the angular frequency $\omega(t)$ [13],

$$\omega^2(t)=\omega_0^2[1+h\ \cos\ \gamma t], \tag{61}$$

where $h\ll 1$ and

$$\gamma=2\omega_0+\epsilon,\quad \epsilon\ll\omega_0\,. \tag{62}$$

In this case it is possible to obtain an analytic approximate expression for the instability exponent χ from the tangent dynamics equation

$$\chi=\frac{1}{2}\left[\left(\frac{h\omega_0}{2}\right)^2-\epsilon^2\right]^{1/2}. \tag{63}$$

From this it follows that parametric resonance occurs when

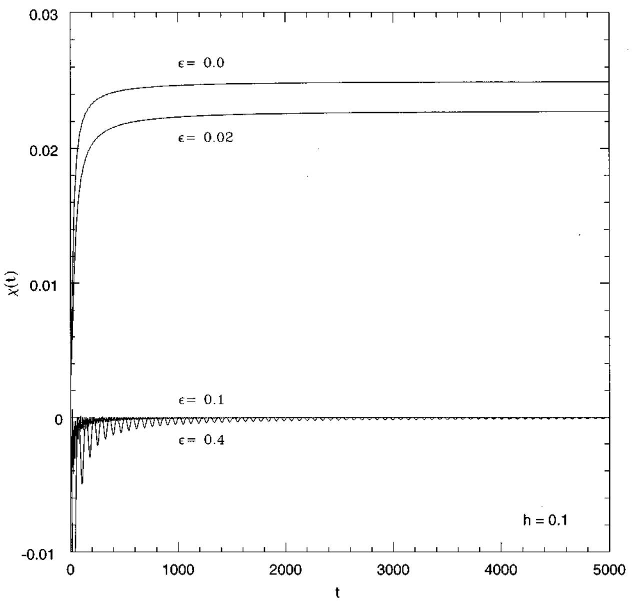

FIG. 1. t-time behavior of the χ_N instability exponent (from the tangent dynamics) for the resonant oscillator in the cases $h=0.1$ and $\epsilon=0.0,0.02,0.1,0.4$. Units are chosen in order to have $\omega_0=1$.

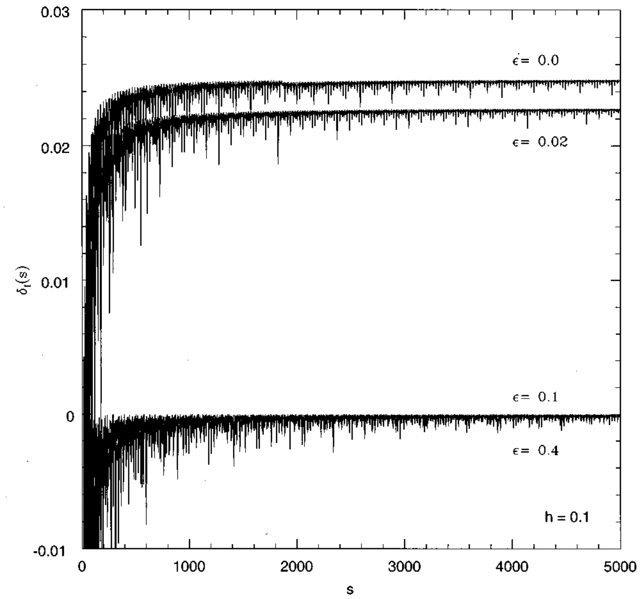

FIG. 2. s-time behavior of δ_I exponents [from Finsler's geodesic deviation equation (59)] for the resonant oscillator in the cases $h=0.1$ and $\epsilon=0.0,0.02,0.1,0.4$. Units are the same as in Fig. 1.

$$|h\omega_0|>|2\epsilon|, \tag{64}$$

when χ is real and positive.

We apply the geometrical method of Finsler spaces to this very simple system and perform a numerical integration of the geodesic equations and of Eq. (59b). In order to have a positive-definite Lagrangian along the trajectory, it is necessary to add to the potential a constant whose magnitude depends on the integration time, which in turn should be long enough to get the asymptotic convergence of the instability exponent. As a check of the integration of the geodesic equations, we also numerically integrate in parallel the tangent dynamics equation and calculate the numerical χ_N exponent, obtaining results in excellent quantitative agreement with Eq. (63). The time behavior of χ_N exponents, for different values of h and ϵ with $\omega_0=1$, is shown in Fig. 1. In Fig. 2 we show the behavior of the numerical δ_I exponents defined in Eq. (52) as functions of the time s, for the same cases of Fig. 1. Both the positive and zero values of δ_I are in good qualitative and quantitative agreement with the χ estimate given by Eq. (63). In Fig. 3 we show the rate of growth of the perturbation $z(s)/z_0$ for an *unstable* orbit with $h=0.1$ and $\epsilon=0.0$ [Fig. 3(a)] and for a *stable* orbit with $h=0.1$ and $\epsilon=0.1$ [Fig. 3(b)]. In the first case the exponential behavior is clear, while in the second the perturbation oscillates around z_0.

In order to test the reliability of the approach, we reconsider here the geometrical interpretation of the mechanism, which generates an exponential divergence of nearby orbits, and compare it with its well-known dynamical interpretation. In all the cases shown above the curvature λ given by Eq. (60) is a positive oscillating function with mean value λ equal to 1 and amplitude equal to $H\simeq h$. Equation (59b) is a Hill equation [17] in which λ is a *positive* time-dependent angular frequency, so it can lead to an exponentially divergent solution if the condition for parametric resonance is satisfied. In fact, writing

$$\lambda(s)=\overline{\lambda}[1+H\cos(\Gamma s)], \tag{65}$$

in which Γ is the angular frequency of $\lambda(s)$, and remembering the resonance condition equation (64), an unstable orbit is characterized by

$$\Gamma=2\overline{\lambda}+E, \tag{66}$$

with $|E|<H\cdot\overline{\lambda}/2=H/2$. For the cases discussed above $H\simeq h=0.1$, so $|E|<0.05$ and it follows that the exponential divergence occurs when

$$2\overline{\lambda}-E<\Gamma<2\overline{\lambda}+E. \tag{67}$$

Setting $f=\Gamma/2\pi$, the condition on the frequency f for parametric resonance is

$$0.310\,35<f<0.326\,27. \tag{68}$$

In Fig. 4 we have shown the spectrum $\Omega(f)$ of $\lambda(s)$ in four cases with $h=0.1$ and $\epsilon=0.0,0.02,0.1,0.4$. The "resonant band" for which relation (68) is fulfilled is also indicated by two vertical lines at 0.310 35 and 0.326 27. From this figure it is clear that an unstable orbit occurs when the spectrum of $\lambda(s)$ shows a maximum in the resonant band.

Summarizing, from the integration of the geodesic deviation we conclude that the geometrical description gives results in perfect agreement with the tangent dynamics, so these results are an intrinsic property of the system and do not depend on the time gauge chosen; the unstable behavior is caused by *parametric resonance* and not by negative values of the curvature λ; and from Fig. 4 it is clear that instability is present when the frequency of λ is twice the average value $\lambda(s)$, thus satisfying the resonance condition.

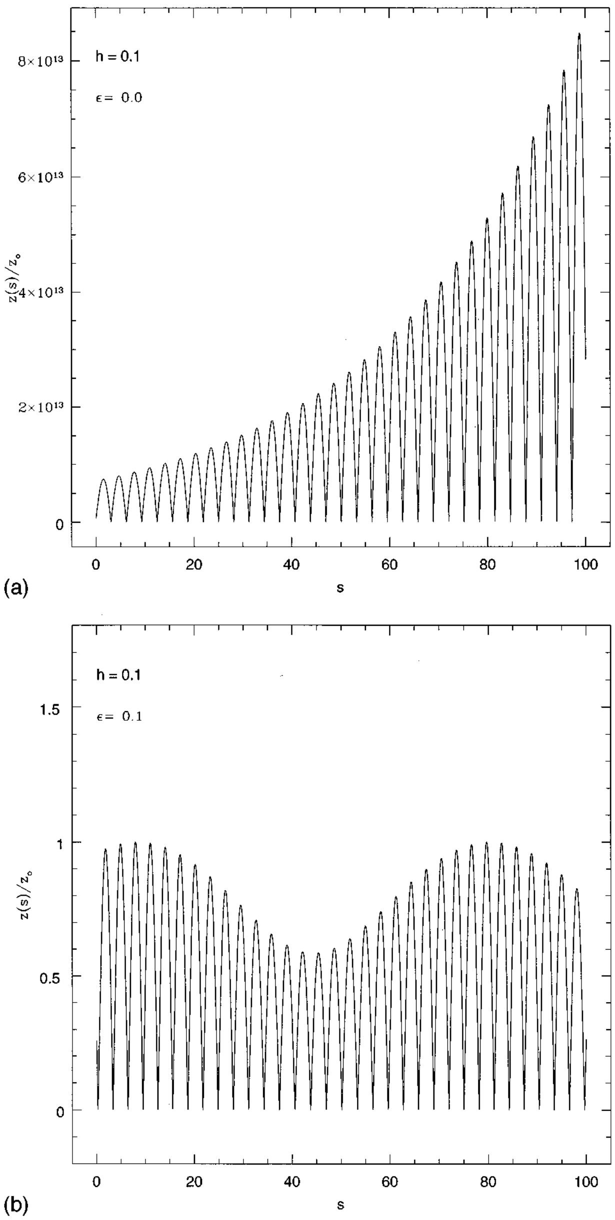

FIG. 3. Evolution of a perturbation compared with its initial value as a function of the affine parameter corresponding to (a) $h=0.1$ and $\epsilon=0.0$ (unstable orbit) and (b) $h=0.1$ and $\epsilon=0.1$ (stable orbit). Units are the same as in Fig. 1.

B. Bianchi IX cosmological models

The controversial outcomes on the possibly chaotic nature of Bianchi IX dynamics have been subject of debate in the past two decades. The source of these discrepancies relies on the choice of the time variable adopted. As it is difficult to establish an *a priori* best variable for this (and a similar) problem, we argue that the use of Finsler geometrization of dynamics is the most appropriate framework in which a more intrinsic description can be worked out. Indeed, as it has been already recalled in the introduction and as it is clearly stated in the (few) textbooks on the topic (see [5a]), one of the main features of the Finsler spaces is related to the requirement of invariance under rescaling of the time parameter. This property makes favorable the Finslerian setting in the discussion of an intrinsic characterization of the qualitative dynamic behavior of relativistic systems, in which the choice of *the time* is highly arbitrary.

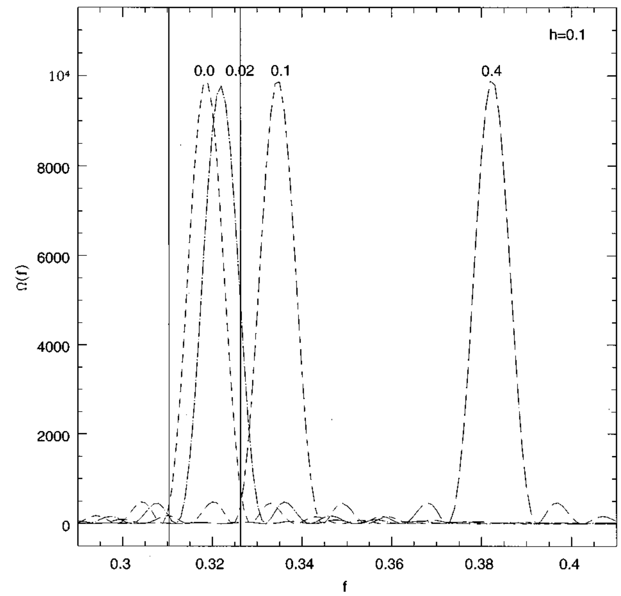

FIG. 4. Spectrum of curvature λ against frequency in several cases. In the figure the lower and the upper limits given by the resonance condition (68) have been indicated. The spectra with a maximum in the resonant band correspond to the unstable orbits of Figs. 1 and 2. Units are the same as in Fig. 1.

So the wide applicability of the Finsler geometrodynamics is strikingly evident in the case of the Bianchi IX models, for which the vanishing Hamiltonian is [7]

$$\mathcal{H}=\frac{(\beta_{+,\tau})^2}{2}+\frac{(\beta_{-,\tau})^2}{2}-\frac{(\alpha_{,\tau})^2}{2}+U(\alpha,\beta_+,\beta_-)=0, \qquad (69)$$

where

$$U=\tfrac{1}{8}e^{4\alpha}\{\tfrac{1}{3}e^{-8\beta_+}-\tfrac{4}{3}e^{-2\beta_+}\cosh(2\sqrt{3}\beta_-)+\tfrac{2}{3}e^{4\beta_+}[\cosh(4\sqrt{3}\beta_-)-1]\}, \qquad (70)$$

β_+, β_-, and α are functions of the scale factors a, b, and c of the mixmaster universe, the derivatives means differentiation with respect to τ ($d\tau=dt/abc$, t is the proper time), and U is the potential of the system. The constraint $\mathcal{H}=0$ is a consequence of the covariance of the general relativity whose field equations are

$$\frac{d^2\alpha}{d\tau^2}=U_{,\alpha},$$

$$\frac{d^2\beta_+}{d\tau^2}=-U_{,\beta_+},$$

$$\frac{d^2\beta_-}{d\tau^2}=-U_{,\beta_-}. \qquad (71)$$

Note that for this system the kinetic part is not positive definite and the related singularities have been the main source of criticism against the use of Jacobi geometrization [9]. The present approach overcomes all these troubles.

We can construct the homogeneous Lagrangian [10,5(b),16]

$$\Lambda=\frac{1}{2\tau'}\,(\alpha'^2-\beta_+'^2-\beta_-'^2)-W(\alpha,\beta_+,\beta_-)\tau', \quad (72)$$

where

$$W(\alpha,\beta_+,\beta_-)=-U(\alpha,\beta_+,\beta_-)+c, \quad (73)$$

in which the gauge is fixed by the choice of the (negative) constant c, added in order to have a positive Lagrangian, so that $\tau'>0$. Note that

$$\tau'=\frac{d\tau}{ds}=\frac{1}{\Lambda\left(x^i,\,\frac{dx^i}{ds}\right)}=\frac{1}{\mathcal{L}}, \quad (74)$$

where

$$\mathcal{L}=-\frac{(\beta_{+,\tau})^2}{2}-\frac{(\beta_{-,\tau})^2}{2}+\frac{(\alpha_{,\tau})^2}{2}-W(\alpha,\beta_+,\beta_-)>0. \quad (75)$$

We evaluate the covariant and the contravariant metric, arriving at the expressions for the geodesic equations, written in terms of the affine parameter s,

$$\beta''_\pm=-\frac{\partial U}{\partial\beta_\pm}\,\tau'^2+\frac{\beta'_\pm}{\tau'}\,\tau'',$$

$$\alpha''=\frac{\partial U}{\partial\alpha}\,\tau'^2+\frac{\alpha'}{\tau'}\,\tau'',$$

$$\tau''=-2\tau'^2\,\frac{dU}{ds}. \quad (76)$$

These equations, when rewritten in terms of the time τ, reduce to Eqs. (71). In addition to Eqs. (71), we have to consider the constraint $\mathcal{H}=0$ [Eq. (69)].

For the geodesic deviation equation, we calculate the trace of the stability tensor

$$\mathrm{tr}H_F=\mathrm{tr}H^i{}_j=\tau'^2\Delta^*U-3\tau'\,\frac{d^2U}{ds^2}+3\tau'^2\left(\frac{dU}{ds}\right)^2, \quad (77)$$

where

$$\Delta^*U=\frac{\partial^2U}{\partial\beta_+^2}+\frac{\partial^2U}{\partial\beta_-^2}-\frac{\partial^2U}{\partial\alpha^2}. \quad (78)$$

In the expression (77) for the trace of the stability tensor there is a positive term and two terms whose sign is not definite. This seems to suggest that, also in this particular dynamical system, the origin of the dynamical instability is not (or not only) related to the negativity of (some) curvature, but rather to the fluctuations of the geometric quantities (see, e.g., [3,4,10,5(b)]). If this is the case, nothing can be said about the relationship between the instability time scale of trajectories and relaxation properties of the system.

Now use the constraint (69) and Eq. (73) for the potential W to reexpress the Lagrangian $\mathcal{L}$ defined by Eq. (75) in terms of

$$\mathcal{L}=2U-c, \quad (79)$$

from which, remembering Eq. (74), it follows that

$$\tau'=\frac{1}{\mathcal{L}}=\frac{1}{2U-c}. \quad (80)$$

The value of constant c is determined in order to have $\tau'>0$ so that the trace of the stability tensor (77) never diverges.

In order to compare the Finsler metric with the Jacobi one, we also calculated the trace of the stability tensor in the Jacobi metric for this dynamical system [3,8,9–10,5(b)]

$$\mathrm{tr}H_J=\frac{1}{2\mathcal{T}^2}\left[\Delta^*U+\frac{(\nabla^*U)^2}{\mathcal{T}}+\frac{1}{2}\left(\frac{dU}{ds}\right)^2+\mathcal{T}\frac{d^2U}{ds^2}\right], \quad (81)$$

where

$$\mathcal{T}=\frac{(\beta_{+,\tau})^2}{2}+\frac{(\beta_{-,\tau})^2}{2}-\frac{(\alpha_{,\tau})^2}{2}. \quad (82)$$

In contrast to Eq. (77), there is now also the term

$$(\nabla^*U)^2=\left(\frac{\partial U}{\partial\beta_+}\right)^2+\left(\frac{\partial U}{\partial\beta_-}\right)^2-\left(\frac{\partial U}{\partial\alpha}\right)^2. \quad (83)$$

It is clear why the Jacobi metric cannot work properly: the trace diverges when the ''kinetic energy'' (or the potential U) vanishes and this happens infinitely many times (and not only on the boundary of the region of motion) going towards the singularity. This does not occur in the Finsler manifold introduced above.

We can better understand this by considering the line element of the Jacobi and Finsler metrics

$$ds_J=-\sqrt{2}U\,d\tau, \quad (84)$$

$$ds_F=\mathcal{L}d\tau=(2U-c)d\tau. \quad (85)$$

Thus, while $ds_J=0$ if $\mathcal{T}=0$ (and this cannot be avoided by adding a constant), this does not happen for ds_F because in this case the conformal factor is the Lagrangian of the system, to which we can add a gauge function without changing the equations of motion.

Finally, we also make a brief comment on the use of the scalar curvature in order to test the instability properties of the geodesic flow. As it is clear from previous works (see, e.g., [3,4,10,5(b)]), the average scalar curvature often has nothing to do with the instability properties of dynamics, except when it is constant and negative. Numerical simulations have been performed in order to estimate averages and fluctuations of geometrical quantities related to the Finslerian transcription of Bianchi IX dynamics and to compare them with the qualitative behavior of its solutions. These results will be presented elsewhere [16].

C. Systems with a potential linearly depending on velocities: The restricted three-body problem

One of the most important applications of the Finsler geometry in the description of the chaotic properties of the

Lagrangian systems is in the case in which the potential depends on both coordinates and velocities, because for these systems the Jacobi and Eisenhart metrics cannot be applied. As an illustrative example we show how the restricted three-body problem can be described in the Finsler geometrical approach. In this case, there are two degrees of freedom and an effective potential given by (in the rotating coordinate system; see [18], p. 242)

$$U_{\mathrm{eff}}=U(x,y)-x\dot{y}+y\dot{x},$$

$$U(x,y)=-\frac{1}{2}\,(x^2+y^2)-\frac{1-\overline{\mu}}{\rho_1}-\frac{\overline{\mu}}{\rho_2}, \tag{86}$$

where $\overline{\mu}$ is the smallest mass [18,19] and

$$\rho_1=\sqrt{(x+\overline{\mu})^2+y^2},\quad \rho_2=\sqrt{(x+\overline{\mu}-1)^2+y^2}. \tag{87}$$

The Lagrangian is given by

$$L=\frac{1}{2}\,(\dot{x}^2+\dot{y}^2)+(x\dot{y}-y\dot{x})-U(x,y), \tag{88}$$

from which one obtains the homogeneous Lagrangian

$$\Lambda=\frac{1}{2t'}\,(x'^2+y'^2)+(xy'-yx')-U^*(x,y)t', \tag{89}$$

where $U^*(x,y)$ is the potential U with the addition of an arbitrary gauge function that gives a positive-definite Lagrangian. Expressions for the covariant components of the metric tensor follow from Eqs. (24)–(27),

$$g_{tt}=\frac{3T^2}{t'^4}+U^{*2}+\frac{2T}{t'^3}\,(-x'y+y'x),$$

$$g_{xx}=\left(\frac{x'}{t'}-y\right)^2+\frac{\Lambda}{t'},$$

$$g_{yy}=\left(\frac{y'}{t'}+x\right)^2+\frac{\Lambda}{t'},$$

$$g_{tx}=-\left(\frac{T}{t'^2}+U^*\right)\left(\frac{x'}{t'}-y\right)-\frac{x'}{t'^2\Lambda},$$

$$g_{ty}=-\left(\frac{T}{t'^2}+U^*\right)\left(\frac{y'}{t'}+x\right)-\frac{y'}{t'^2\Lambda},$$

$$g_{xy}=\left(\frac{x'}{t'}-y\right)\left(\frac{y'}{t'}+x\right), \tag{90}$$

where $T=(x'^2+y'^2)/2$. Evaluating expressions (33) and (34) leads to

$$A=\frac{dU^*}{ds}+xx'+yy'-\frac{t'}{2}\,(U^*_{,x}y-U^*_{,y}x),$$

$$B=\frac{dA}{ds}+t'A^2, \tag{91}$$

so that the eigenvalues of Eq. (46) evaluate to

$$\lambda_{(\alpha)}=Bt'+(t')^2+(t')^2\mu_{(\alpha)},\quad \alpha=1,2, \tag{92}$$

where $\mu_{(\alpha)}$ are the eigenvalues of the Hessian matrix of the potential. The trace of the stability tensor is

$$H^i{}_i=t'\Delta U^*+2t'\left[\frac{dA}{ds}+t'A^2\right]+2t'^2, \tag{93}$$

while the covariant components are given by

$$H_{tt}=2\,\frac{T}{(t')^2}\,(B+t')+\frac{1}{t'}\,[U^*_{,xx}(x')^2+U^*_{,yy}(y')^2+2U^*_{,xy}x'y'],$$

$$H_{\alpha\alpha}=B+t'+t'U^*_{,\alpha\alpha},$$

$$H_{t\alpha}=-\frac{x'^{\alpha}}{t'}\,(B+t')-\frac{dU^*_{,\alpha}}{ds},$$

$$H_{xy}=t'U^*_{,xy}. \tag{94}$$

In this way the dynamics is reduced to a geodesic flow on a Finsler manifold. The numerical results relevant to the chaotic nature of this system are presented elsewhere [15]

V. CONCLUSIONS AND FUTURE WORK

In this paper we have developed a formalism suited to extending the geometrical approach to the description of chaos in a class of manifolds larger than the Riemannian one and as a consequence for more general dynamical systems. In this wider setting we have evaluated the quantities involved in the geodesic deviation equation, which is the natural tool for the study of the stability properties of the geodesic flow. The greater generality of Finsler geometry with respect to the Riemannian one manifests itself in allowing the description of a wider class of Lagrangian systems, including those with an indefinite kinetic structure, as those coming from the theory of general relativity. For such a class of dynamical systems the geometrical description is particularly helpful because it provides a natural way to introduce quantities with an intrinsic meaning that are well suited to describing the chaotic or regular properties of the motions of the system, i.e., not depending on the choice of coordinates adopted.

The need and the importance of a gauge-independent picture is indispensable in the case of the Bianchi IX cosmological models, for which many authors [7,9] have explicitly attributed the origin of the controversial results on the nature of their dynamics to the time-gauge-dependent methods and criteria adopted. The results of our studies on the topic in the framework of the Finsler geometry will appear elsewhere [16].

The wider applicability of Finsler geometry even in classical mechanics also relies on the nature of the conformal factor, which is nothing but a Lagrangian of the dynamical system, and this allows one to overcome the main shortcoming of the Jacobi metric for systems with a few degrees of freedom or described by an indefinite kinetic structure. Indeed, while the singularities of the Jacobi metric on (or inside) the boundary of the region of allowable motions cannot be avoided, the gauge freedom allows one to add to the Lagrangian a total time derivative $d\mathcal{G}/dt$, which cancels any possible singularity, leaving unchanged the equations of motion for the system. This further advantage of the Finsler approach is important since the chance of a vanishing kinetic energy is not negligible, and this is clearly the case for classical dynamical systems with a few degrees of freedom as well as for relativistic dynamical systems, whose kinetic part is indefinite.

Finally, we remark that the analysis presented here is a generalization of the geometrical approach to dynamics pursued in previous works [3,4], whose goal was the attempt to find a relationship between the occurrence of chaotic dynamics and the curvature properties of the underlying manifold. We have shown how the analysis can be carried out for a simple and well understood dynamical system whose chaotic behavior is due to parametric resonance. In a series of forthcoming papers [14–16], we plan to show that the mechanism responsible for the onset of chaos is usually very subtle, even for systems with a few (two or three) degrees of freedom, and is always related to the properties of the fluctuating curvatures around their average positive values, only marginally to the frequency of occurrence of negative values, and never to an average negative value. When the number of degrees of freedom is small, the analysis gets increasingly more involved as the number increases, and only when it becomes very large can some simplifying assumptions be made [3,4,10,5(b)], based on statistical considerations (essentially on the central limit theorem).

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